April 2025

Easier

On the M25, how much longer is the slow lane going clockwise than the slow lane going anticlockwise?

The M25 is, for the most, part a dual 4-lane motorway. There are still a few 3-lane sections near junctions, and there are a couple of 5-lane sections on the west near Heathrow. Those that know the M25 well will know that there is a 1km section just east of the A1M where the two lanes separate – this extends the counter-clockwise route by about 15m compared to the clockwise route, but for our purposes we'll ignore this detail. The other thing we'll need is the width of a lane on a motorway – which is about 12ft or 3.65m. The central reservation will be taken to be about the width of a single lane.

Given the above we can say that on average the separation of the two slow lanes is 8 lane-widths or roughly 30m.

If we model the inner lane of the M25 as a circle of radius R the difference in length between the two lanes is:

$$\underbrace{2\pi(R+30m)}_{clockwise} - \underbrace{2\pi R}_{countre-clockwise} = 2\pi.30m \approx 190m$$

Note that this doesn't depend on the value of R – all the answer depends on is the separation of the two lanes. If the M25 were circular, the difference in length would grow in proportion to the fraction of a complete rotation that has been made so far; i.e. the difference in length is proportional to the separation and the angle through which the road has turned.

Harder

A smooth curve on a flat piece of paper has a small but finite width. Find a formula for the difference in length between the longer two sides of the curve. Use this to find a formula for the radius of curvature of the line at any point.

The first thing to do is to parameterise the curve in some way, and we'll do this by imagining walking along the curve and labelling the points on the midpoint of the line by their Cartesian coordinates as a function of time. We can choose these labels so that the time of the start of the curve is 0 and at the end is 1. Hence the centre-line is at

$$\big(x(t), y(t)\big) \quad \text{where } 0 \le t \le 1$$

although from now on we'll drop the explicit time argument. In order to find the difference in length of the two sides we need to know where the sides are. To do this we need to find the direction of the curve, construct a unit vector at right angles to this, and move a distance δ, i.e. half the line-width, in that direction.

The direction of the curve is our velocity vector, i.e.

$$direction\ of\ travel = (\dot{x}, \dot{y})$$

A unit vector perpendicular to this is

$$unit\ perpendicular = \frac{(-\dot{y}, \dot{x})}{\sqrt{\dot{x}^2 + \dot{y}^2}}$$

The position of one of the line edges is therefore

$$(x, y) + \delta \frac{(-\dot{y}, \dot{x})}{\sqrt{\dot{x}^2 + \dot{y}^2}} = \left(x - \delta \frac{\dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}}, y + \delta \frac{\dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2}}\right)$$

To find the length of the line edge we can integrate the *speed* at which the edge points are moving. The *velocity* is, of course, the rate of change of the position:

$$velocity = \left(\dot{x} - \delta \frac{\ddot{y}}{(\dot{x}^2 + \dot{y}^2)^{1/2}} + \delta \frac{\dot{y}(\dot{x}\ddot{x} + \dot{y}\ddot{y})}{(\dot{x}^2 + \dot{y}^2)^{3/2}}, \dot{y} + \delta \frac{\ddot{x}}{(\dot{x}^2 + \dot{y}^2)^{1/2}} - \delta \frac{\dot{x}(\dot{x}\ddot{x} + \dot{y}\ddot{y})}{(\dot{x}^2 + \dot{y}^2)^{3/2}}\right)$$

The speed is the magnitude of the velocity vector, and after a small amount of algebra we find:

$$speed = \left(\dot{x}^2 + \dot{y}^2 - 2\delta \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{(\dot{x}^2 + \dot{y}^2)^{1/2}} + \delta^2 \frac{\ddot{x}^2 + \ddot{y}^2}{\dot{x}^2 + \dot{y}^2}\right)^{\frac{1}{2}}$$

As the width is small, we can ignore terms that are of order δ^2, so

$$speed = (\dot{x}^2 + \dot{y}^2)^{1/2} - \delta \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{\dot{x}^2 + \dot{y}^2}$$

and the length of this edge in total is the integral over time:

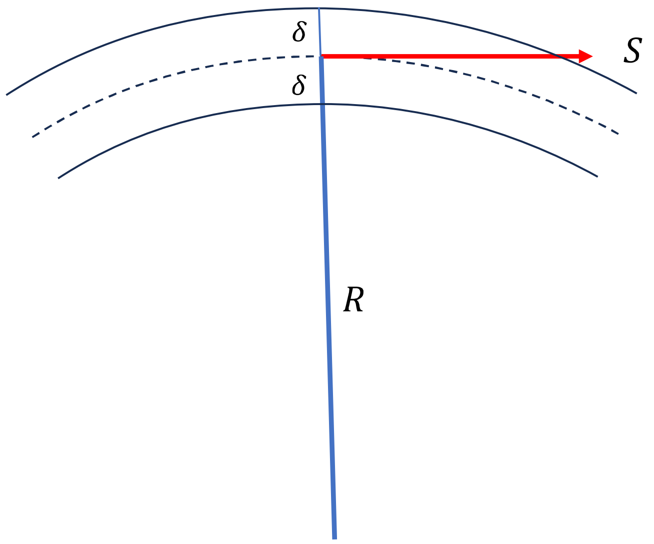
$$length\ of\ edge = \int_0^1 (\dot{x}^2 + \dot{y}^2)^{1/2}\, dt - \delta \int_0^1 \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{\dot{x}^2 + \dot{y}^2}\, dt$$

The problem asks us to find the difference in the length of the two edges, which is

$$2\delta \int_0^1 \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{\dot{x}^2 + \dot{y}^2}\, dt$$

where 2δ is the width of the curve.

The second part of the question asks us to relate this to the radius of curvature. Call the radius of curvature is R and the speed along the centre-line is S:



The rates of change of the lengths of the edges are

$$\underbrace{(R \pm \delta)}_{radius} \quad \underbrace{\frac{S}{R}}_{angular\ rate}$$

so the difference in edge lengths grows at a rate

$$2\delta\frac{S}{R}$$

Comparing this with the formula we have already found for the relative change in edge length we see that

$$\frac{S}{R}=\frac{\dot{x}\ddot{y}-\dot{y}\ddot{x}}{\dot{x}^2+\dot{y}^2}$$

Finally, by noting that the speed along the centre-line is $S=\sqrt{\dot{x}^2+\dot{y}^2}$, we find

$$\frac{1}{R}=\frac{\dot{x}\ddot{y}-\dot{y}\ddot{x}}{(\dot{x}^2+\dot{y}^2)^{3/2}}$$

As a matter of interest, this formula was first discovered by Newton in about 1664 when he was investigating the "crookednesse" of plane curves.

Not yet used

> Prove Fermat's little theorem which states that for any integer a and prime p:
> $a^p - a$ is an integer multiple of p.

There are a number of proofs of Fermat's little theorem as a quick browse through Wikipedia soon shows. The solution that occurred to me was based on induction which, in my humble opinion, is the easiest to understand and so I'll give it here:

Assume the theorem is true for some integer a; this means

$$a^p - a = p.L$$

for some integer L. Now consider $a + 1$:

$$(a+1)^p = \sum_{n=0\dots p} a^n \frac{p!}{n!\,(p-n)!}$$

The numerator in the binomial term contains a factor of p, which is prime. This cannot be cancelled by a factor in the denominator unless n is 0 or p, as otherwise the factors are all less than p. This means that

$$(a+1)^p = a^p + p.K + 1$$

for some integer K. We therefore have

$$(a+1)^p - (a+1) = (a^p + p.K + 1) - (a+1)$$

The 1s on the right cancel immediately giving

$$(a+1)^P - (a+1) = (a^p - a) + p.K$$

and by assumption

$$a^p - a = p.L$$

Hence

$$(a+1)^P - (a+1) = p(L+K)$$

Which tells us that if the theorem is true for a it is true for $a + 1$. But the theorem is trivially true for $a = 0$, so the theorem is true for all positive a.

To show the theorem is true for negative integers, b, we can write $a = -b$, to discover that

$$(-b)^p - (-b) = p.L$$

If p is an odd prime the factor of $(-1)^p$ is -1; this immediately gives us

$$b^p - b = p.(-L)$$

This only leaves $p = 2$, but in this case we have, $b^2 - b$ on the left hand side, and this factorises into $b(b-1)$. One of these factors must be even, i.e. a multiple of p, thereby proving the theorem.

Multinomial coefficients

In how many distinct ways can you arrange n_A letter As, n_B letter Bs, n_C Cs, etc?

For convenience let $N = n_A + n_B + \cdots$.

If we had N different letters then there would be $N!$ ways to arrange them. So if we were to add a distinguishing mark to the letters in our example, e.g. by adding a subscript, then this tells us that there are $N!$ ways to arrange them.

Now let us make a big list of all the permutations of the letters with their subscripts attached, and let us first focus on all of the As, i.e. A_1through to A_{n_A}. For every permutation where we have the As appearing in subscript order at a given set of positions, e.g.

$$*** A_1 **** A_2 * A_3 ****** \dots *** A_{n_A} *$$

we will also find in the big list examples where everything is the same *apart from the fact that the subscripts on the As are different*. As they must all be in the list somewhere we will eventually find every permutation of the subscripts, and there are $n_A!$ such permutations. If we now remove the subscripts on all of the As in the big list we see that every entry will be repeated $n_A!$ times. Removing the duplicates gives us

$$Size\ of\ list\ without\ subscripts\ on\ the\ As = \frac{N!}{n_A!}$$

We now repeat the process of removing the duplicates when the subscripts on the Bs are removed giving

$$Size\ of\ list\ without\ subscripts\ on\ the\ As\ or\ Bs = \frac{N!}{n_A!\, n_B!}$$

And after removing all the subscripts the length of the list - and hence the number of ways to rearrange the letters - will be

$$\frac{N!}{n_A!\, n_B!\, n_C! \dots}$$

<u>Multinomial theorem</u>

Perhaps the commonest application of the formula is in the multinomial theorem – the extension to the binomial theorem:

$$(a + b + c \dots)^N = \sum_{\substack{n_a, n_b, n_c \dots \\ n_A + n_B + \cdots = N}} a^{n_a} b^{n_b} c^{n_c} \dots \frac{N!}{n_a!\, n_b!\, n_c! \dots}$$

This formula can be related to the previous argument by expanding out the left hand whilst keeping the order of the terms and without doing any of the additions. At the risk of being very boring, consider $(a+b+c)^4$:

$(a+b+c)^4 = (a+b+c)(a+b+c)(a+b+c)(a+b+c)$

$=(aa+ab+ac+ba+bb+bc+ca+cb+cc)(a+b+c)(a+b+c)$

$=(aaa+aab+aac+aba+abb+abc+aca+acb+acc+baa+bab+bac+bba+bbb+bbc+bca+bcb+bcc+caa+cab+cac+cba+cbb+cbc+cca+ccb+ccc)(a+b+c)$

$=aaaa+aaab+aaac+aaba+aabb+aabc+aaca+aacb+aacc+abaa+abab+abac+abba+abbb+abbc+abca+abcb+abcc+acaa+acab+acac+acba+acbb+acbc+acca+accb+accc+baaa+baab+baac+baba+babb+babc+baca+bacb+bacc+bbaa+bbab+bbac+bbba+bbbb+bbbc+bbca+bbcb+bbcc+bcaa+bcab+bcac+bcba+bcbb+bcbc+bcca+bccb+bccc+caaa+caab+caac+caba+cabb+cabc+caca+cacb+cacc+cbaa+cbab+cbac+cbba+cbbb+cbbc+cbca+cbcb+cbcc+ccaa+ccab+ccac+ccba+ccbb+ccbc+ccca+cccb+cccc$

Now highlight all the terms containing 2 as 1 b and 1 c:

$aaaa+aaab+aaac+aaba+aabb+aabc+aaca+aacb+aacc+abaa+abab+abac+abba+abbb+abbc+abca+abcb+abcc+acaa+acab+acac+acba+acbb+acbc+acca+accb+accc+baaa+baab+baac+baba+babb+babc+baca+bacb+bacc+bbaa+bbab+bbac+bbba+bbbb+bbbc+bbca+bbcb+bbcc+bcaa+bcab+bcac+bcba+bcbb+bcbc+bcca+bccb+bccc+caaa+caab+caac+caba+cabb+cabc+caca+cacb+cacc+cbaa+cbab+cbac+cbba+cbbb+cbbc+cbca+cbcb+cbcc+ccaa+ccab+ccac+ccba+ccbb+ccbc+ccca+cccb+cccc$

The highlighted terms are an exhaustive list for this combination of letters. (To prove this write 0 for a, 1 for b and 2 for c. The overall list is then just counting in ternery: each term is unique, there are no repeats and there are 3^4 terms in total.)

The number of highlighted terms is precisely the number of ways you can reorder the letters, which we have seen is

$$\frac{N!}{n_A!\,n_B!\,n_C!\ldots} = \frac{4!}{2!\,1!\,1!} = 12$$

The expansion therefore contains the term

$$12a^2bc$$

The multinomial formula just is the sum over all possible combinations of four letters, not just the one we have looked at in detail.