December 2024

Easier

If a mountain gets too tall the energy released by the mountain sinking an inch or two can be enough to melt the rock at the base of the mountain.

Find a formula for the height of the tallest mountains on a planet. Make a crude estimate the size of the smallest possible planet.

Newtons Gravitational constant = 6.7×10^{-11} $m^3.kg^{-1}.s^{-2}$
Rock density = 4000 $kg.m^{-3}$
Latent heat of melting for rock = 100 $kJ.kg^{-1}$

As we only have to make a crude estimate we don't have to be too accurate. We will assume that the planet and the mountain are made of the same material. If the planet has a radius, R, the mass of the planet, M, will be

$$M = \frac{4\pi}{3}\rho R^3$$

where ρ is the density of the rock. The acceleration at the surface of the planet is then

$$g = \frac{GM}{R^2} = \frac{4\pi}{3}\rho G R$$

Consider a small core from the top of the mountain down to the base, call the cross-sectional area A and the height h. The total mass of this core is $\rho A h$, so the energy released when this drops a distance d is force x distance, i.e.

$$energy = \rho A h.\, g.\, d$$

We are told that is this energy in only just enough to melt the rock at the bottom of the mountain then we have determined the maximum height of a mountain. The energy required to melt the rock is the mass of the bit of rock that needs to be melted (this has a cross section A and a height d) times the latent heat of melting:

$$energy = \rho Ad.C$$

Equating these two energies we find

$$\rho Ah.g.d = \rho Ad.C$$

i.e.

$$gh = C$$

hence

$$h = \frac{C}{g} = \frac{3C}{4\pi\rho GR}$$

Substituting the values given leads to

$$h = \frac{3C}{4\pi\rho GR} = \frac{8.9 \times 10^{11} m^2}{R}$$

For the Earth ($R = 6400km$) this estimate is approximately 14 km – which isn't far off the height of Everest.

As the size of the planet decreases there comes a point when the mountain becomes as high as the planet, at which point the object is more asteroid than planet. This happens when

$$R = \frac{8.9 \times 10^{11} m^2}{R}$$

i.e. roughly 1000km – which is about the size of Pluto.

Harder

A Foucault pendulum that has been set swinging North-South will slowly and smoothly rotate to some other orientation. Explain this behaviour using Lagrangian mechanics and note any other interesting aspects of the motion.

The obvious variables to choose to define the problem are oscillations in two orthogonal directions: North-South and East-West. Writing θ_N for the first and θ_E for the other the potential energy is

$$PE = mgL\left(1 - \cos\sqrt{{\theta_N}^2 + {\theta_E}^2}\right)$$

The kinetic energy from the N-S component is reasonably simple, although the fact that the Earths rotation can provide a contribution needs to be remembered. The motion in the E-W direction needs to include the Earth's rotation:

$$KE = \frac{m}{2}\underbrace{\left(L\dot{\theta_N} - \Omega L\theta_E \sin\varphi\right)^2}_{N-S} + \frac{m}{2}\underbrace{\left(L\dot{\theta_E} + \Omega(R+L)\cos\varphi - \Omega L\cos(\varphi - \theta_N)\right)^2}_{E-W} + \frac{m}{2}\underbrace{(\Omega L\theta_E\cos\varphi)^2}_{up-down}$$

Where Ω is the angular rotation rate of the Earth, and R is the radius of the Earth. One of the first things that can be said is that the total energy is not minimised at $\theta_N = 0$. This is because the rotation of the Earth swings the pendulum out slightly. The angle is, however, quite small (a fraction of a degree).

It is also clear that the mass of the pendulum bob is common to all of the terms and so can be removed.

Using a small angle approximation that ignores terms above θ^2 we have

$$PE = gL\left(\frac{{\theta_N}^2}{2} + \frac{{\theta_E}^2}{2}\right)$$

$$KE = \frac{1}{2}\left(L\dot{\theta_N} - \Omega L\theta_E\sin\varphi\right)^2 + \frac{1}{2}\left(L\dot{\theta_E} + \Omega R\cos\varphi + \Omega L\cos\varphi\frac{{\theta_N}^2}{2} + \Omega L\sin\varphi\,\theta_N\right)^2 + \frac{1}{2}(\Omega L\theta_E\cos\varphi)^2$$

The Lagrangian, L, is the difference $KE - PE$, and the equation of motion is, for an angle θ is

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = \frac{\partial L}{\partial \theta}$$

The equations for θ_N and θ_E are therefore (after some algebra)

$$\ddot{\theta_N} - 2\Omega \sin\varphi\, \dot{\theta_E} = \Omega^2\, R/L \cos\varphi \sin\varphi - (g/L - \Omega^2\, R/L \cos\varphi - \Omega^2 \sin^2\varphi)\theta_N$$
$$\ddot{\theta_E} + 2\Omega \sin\varphi\, \dot{\theta_N} = -\, g/L\, \theta_E + \Omega^2 \theta_E$$

By adding a small offset to the N-S component we can simplify slightly the equation of motion for the N-S component. This is the small amount the pendulum swings out because of the Earth's rotation. Writing

$$\theta'_N = \theta_N - \frac{\Omega^2 R \cos\varphi \sin\varphi}{g - \Omega^2 R \cos\varphi - \Omega^2 L \sin^2\varphi}$$

yields, after a little algebra,

$$\ddot{\theta'}_N - 2\Omega \sin\varphi\, \dot{\theta_E} = -\left(\frac{g}{L} - \frac{R}{L}\Omega^2 \cos\varphi - \Omega^2 \sin^2\varphi\right)\theta'_N$$
$$\ddot{\theta_E} + 2\Omega \sin\varphi\, \dot{\theta'}_N = -\left(\frac{g}{L} - \Omega^2\right)\theta_E$$

The left hand side can be written in terms of a matrix operator:

$$D = \begin{pmatrix} \frac{d}{dt} & -\Omega \sin\varphi \\ +\Omega \sin\varphi & \frac{d}{dt} \end{pmatrix}$$

as

$$D^2 \begin{pmatrix} \theta'_N \\ \theta_E \end{pmatrix} = -\frac{1}{L}\begin{pmatrix} g - \Omega^2 R \cos\varphi & 0 \\ 0 & g - \Omega^2 L \cos^2\varphi \end{pmatrix}\begin{pmatrix} \theta'_N \\ \theta_E \end{pmatrix}$$

The first thing to note is that the motion is not simple on long timescales because the values on the right hand side are not the same. This ultimately means the motion cannot be described by a simple linear pendulum motion that is rotating. The motion will change from a linear pendulum motion to a circular motion and back again

The terms proportional to Ω^2 are, however, small and if we ignore them we have

$$D^2 \begin{pmatrix} \theta'_N \\ \theta_E \end{pmatrix} = -\frac{g}{L}\begin{pmatrix} \theta'_N \\ \theta_E \end{pmatrix}$$

This can be factorised to give

$$D \begin{pmatrix} \theta'_N \\ \theta_E \end{pmatrix} = \pm i\sqrt{\frac{g}{L}}\begin{pmatrix} \theta'_N \\ \theta_E \end{pmatrix}$$

This can be written as

$$\frac{d}{dt}\begin{pmatrix}\theta'_N\\ \theta_E\end{pmatrix} = \left[\pm i\sqrt{\frac{g}{L}}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + \Omega\sin\varphi\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\right]\begin{pmatrix}\theta'_N\\ \theta_E\end{pmatrix}$$

The two matrices on the right commute, so the solution can be written as the product of two exponentials, the second of which is a rotation:

$$\begin{pmatrix}\theta'_N\\ \theta_E\end{pmatrix} = exp\left(\pm i\sqrt{\frac{g}{L}}t\right)\begin{pmatrix}\cos(\Omega\sin\varphi\, t) & \sin(\Omega\sin\varphi\, t)\\ -\sin(\Omega\sin\varphi\, t) & \cos(\Omega\sin\varphi\, t)\end{pmatrix}\begin{pmatrix}A\\ B\end{pmatrix}$$

Where A and B are constants. This is the usual Foucault pendulum motion.