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Easier

The Maths and Physics group has decided to visit Mars. Initial estimates put the total payload at 40 tonnes.

Assuming we use liquid hydrogen and oxygen fuel, estimate the minimum rate in kg/s at which we have to burn the fuel just to get the payload off the ground.

From the acceleration due to gravity, g, and the radius of the Earth, R_E, find the escape velocity we will need to achieve.

$g = 10\ ms^{-2}$; $R_E = 6.4$ Mm; $2H_2 + O_2$ releases 12.5 MJ/kg

Minimum fuel rate

The fuel produces 12.5 MJ/kg and this is going to appear in the exhaust gases. If we burn 1kg of fuel we will liberate 12.5MJ of energy. Using the formula for kinetic energy, $E = \frac{1}{2}mv_{gas}^2$, we find, for 1kg of fuel:

$$12.5 \times 10^6 = \frac{1}{2} \times 1 \times v_{gas}^2$$

which tells us $v_{gas} = 5{,}000$ ms^{-1}. Assuming the gases end up travelling in a straight line behind the rocket the momentum in this 1kg of gas will be 5,000 kgms^{-1}.

If we are burning fuel at a rate of μ kgs^{-1} the force, F, (i.e. the change in momentum per second of the fuel) is:

$$F = \mu v_{gas}$$

or $F = 5000\mu$ kgms^{-2}.

The minimum rate of fuel needed to get the payload off the ground is that which just counteracts the force of gravity on the payload. (You might want to add in the weight of the fuel, but if the objective is "*just* to get the payload off the ground" we can do that for a negligible amount of time and hence need a negligible amount of fuel.) The minimal conditional gives us

$$\mu \times 5000 = g \times 40 \; tonnes = g \times 40{,}000 \; kg$$

and this leads to

$$\mu = 80 \text{ kgs}^{-1}$$

Of course in order to get into space we will need a lot more than this, probably more than an order of magnitude more, as most space rockets are mostly fuel when they are on the ground.

Escape velocity

The force of gravity obeys an inverse square law, and at the surface of the Earth this produces an acceleration equal to g, so

$$\frac{M_E G}{R_E^2} = g$$

The potential energy of a mass m at a distance r from the centre of the Earth is

$$\frac{mM_E G}{r} = \frac{mgR_E^2}{r}$$

To reach escape velocity our mass will have to be given just enough kinetic energy to overcome this potential energy, so, as we are starting from the surface of the Earth, i.e. $r = R_E$, we must have

$$\frac{1}{2} mV^2 = \frac{mgR_E^2}{R_E} = mgR_E$$

i.e. the escape velocity is

$$V = \sqrt{2gR_E}$$

Plugging in the numbers gives V approximately equal to 11.3 kms^{-1}.

Harder

Calculate the minimum mass of fuel we'll need, and update the minimum fuel rate.

Make some more realistic assumptions concerning your design and update your calculations.

Compare your rocket with Saturn V which could take 40 tonnes to the Moon. [Saturn V had a dry weight (including payload) of ~225 tonnes and used ~2,700 tonnes of fuel.]

Minimum mass of fuel

To find the minimum mass of fuel we will have to solve, at least partially, the equation of motion.

We know from the easier problem that the force from the engine is

$$F = \mu v_{gas}$$

And we also know that this has to overcome the weight of the rocket and provide the upward acceleration, i.e.

$$F = \mu v_{gas} = \left(M + m(t)\right)\left(\frac{gR_E^2}{\left(R_E + h(t)\right)^2} + a(t)\right)$$

Where M is the mass of the payload, $m(t)$ is the mass of the fuel, $h(t)$ is the height of the rocket and $a(t)$ its acceleration.

The rate at which fuel is used, μ, is the rate of *decrease* of the mass of fuel, so

$$\mu = -\dot{m}(t)$$

and the acceleration is the second derivative of the height, giving us:

$$-\dot{m}(t)v_{gas} = \left(M + m(t)\right)\left(\frac{gR_E^2}{\left(R_E + h(t)\right)^2} + \ddot{h}(t)\right)$$

As the mass of the payload doesn't change, we can write this as

$$-\frac{\left(M+\dot{m}(t)\right)}{\left(M+m(t)\right)}v_{gas}=\frac{gR_E^2}{\left(R_E\ +h(t)\right)^2}+\ddot{h}(t)$$

And as the velocity of the exhaust gas doesn't change with time either we can integrate this to find the vertical velocity, $\dot{h}(t)$,

$$log\left(\frac{M+m(0)}{M+m(t)}\right)v_{gas}=\int_0^t\frac{gR_E^2}{\left(R_E\ +h(t)\right)^2}dt+\dot{h}(t)$$

The first term on the right can be made small by accelerating quickly, so to estimate the minimum mass of fuel we could ignore that term. The aim is to get into space, so by the time we run out of fuel (i.e. $m(t)=0$) we need the vertical velocity to be the escape velocity ($\dot{h}(t)=V$)

$$log\left(1+\frac{m(0)}{M}\right)>\frac{V}{v_{gas}}$$

With V = 11.3 kms^{-1} and v_{gas} = 5 kms^{-1} we have a minimum fuel mass of

$$m(0)=8.6\times M$$

The minimum fuel rate would then be ~690kgs^{-1} - which is a lot of fuel!

More realistic assumptions

The first problem is that, in our calculation, the rocket only consists of a payload and fuel, when in fact the fuel tanks, engines and control equipment will themselves weigh a considerable amount. If we assume the control is of a similar mass to the payload and the tanks and engines are a fraction, say between 5 and 10%, the mass of the fuel we get:

For 5%: $m(0)=8.6\times(2M+m(0)/20)$; or $m(0)\approx 30M\approx 1{,}200$ tonnes

For 10%: $m(0)=8.6\times(2M+m(0)/10)$; or $m(0)\approx 125M\approx 5{,}000$ tonnes

Comparison with Saturn V

Saturn V, which was specified to take 40 tonnes to the Moon, used 2,700 tonnes of fuel, so the fuel was therefore $67.5M$ which is roughly in line with our guess. The dry mass excluding the payload was 185 tonnes, and if we subtract off our guess for the control (i.e. M or 40 tonnes) the tanks and engines would have been 145 tonnes or 5.4% of the mass of fuel – which is towards the lower end of our estimate.