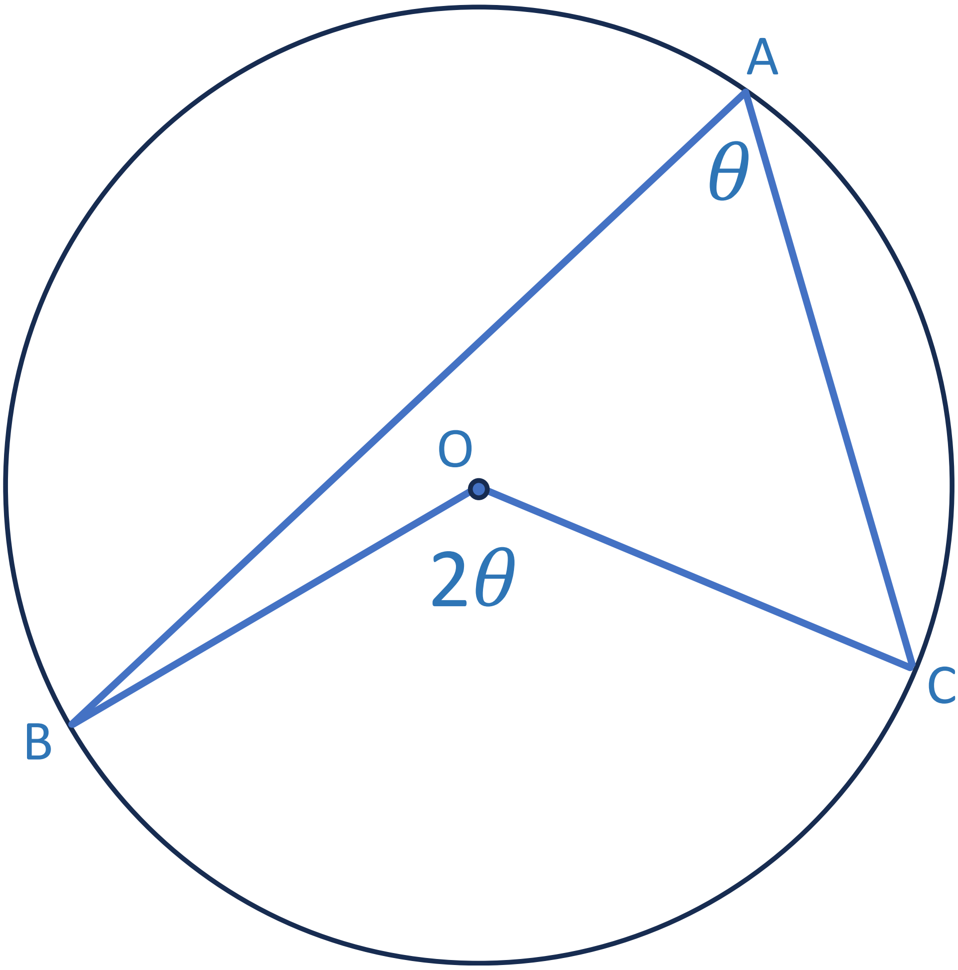
January 2024

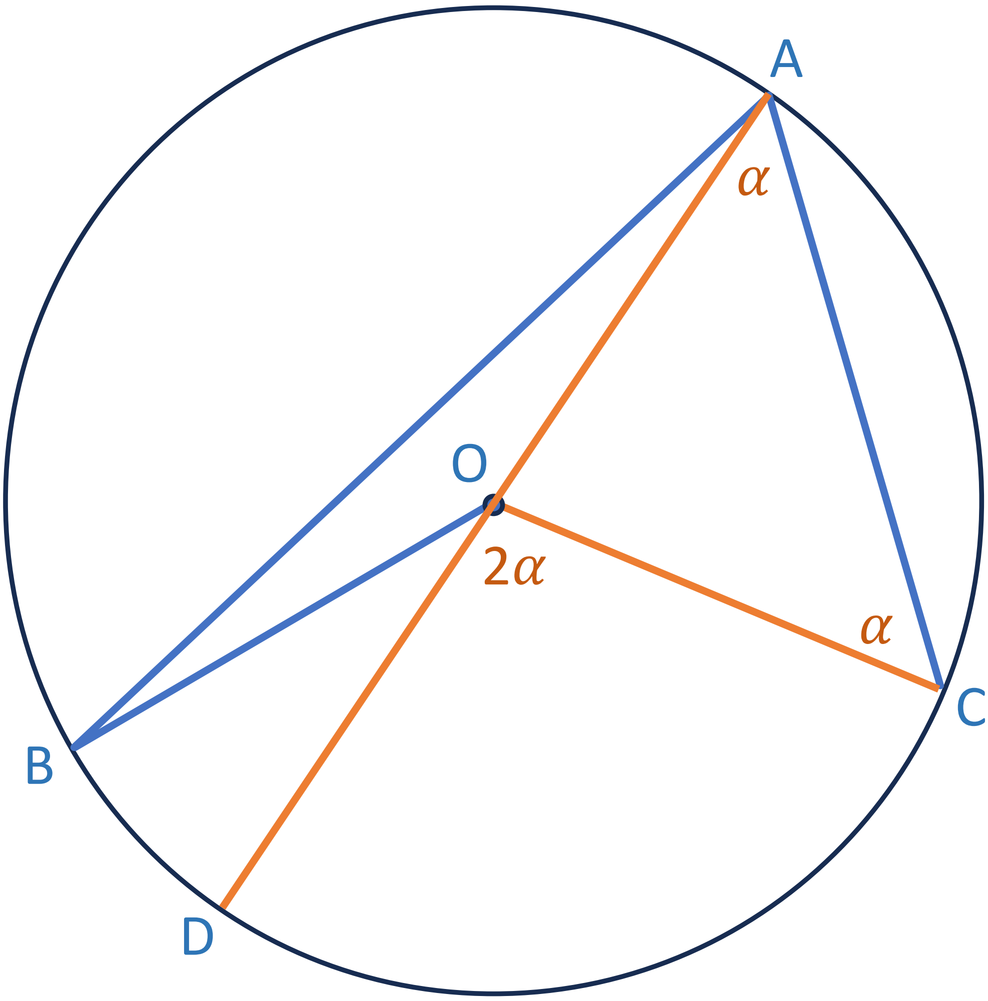
Easier

Prove the angle subtended by an arc at the centre of a circle is twice the angle subtended at the circumference.



First, construct the diameter between the two marked angles. In the diagram below this is the line AOD.

If we now concentrate on the triangle AOC we note that OA and OC are radii of the circle, so AOC is isosceles. This means ∠OAC equals OCA, and is marked as α in the diagram. The angle COD is the exterior angle which is the sum of the two angles opposite, so ∠COD is 2α, i.e. it is twice ∠OAC.



A similar argument for triangle AOB tells us that ∠BOD is twice ∠OAB.

The sum of the angles from these two triangles gives the required result *in this case*.

There are actually two more cases to consider:

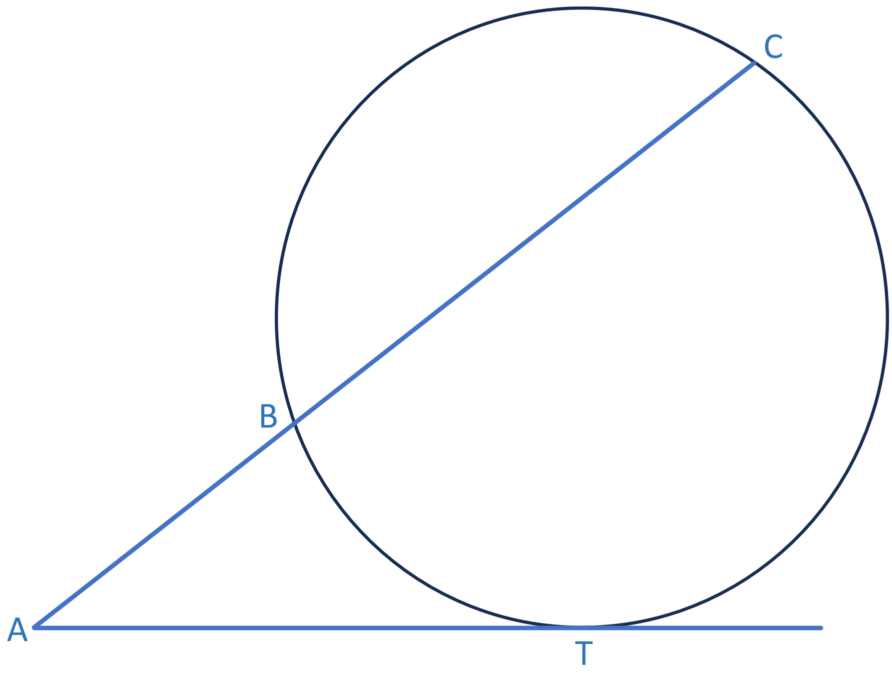
— when B (or C) coincides with D: in which case we only need to consider one of the triangles above; and

— when B lies between D and C: in which case we must subtract the two exterior angles rather than adding them;

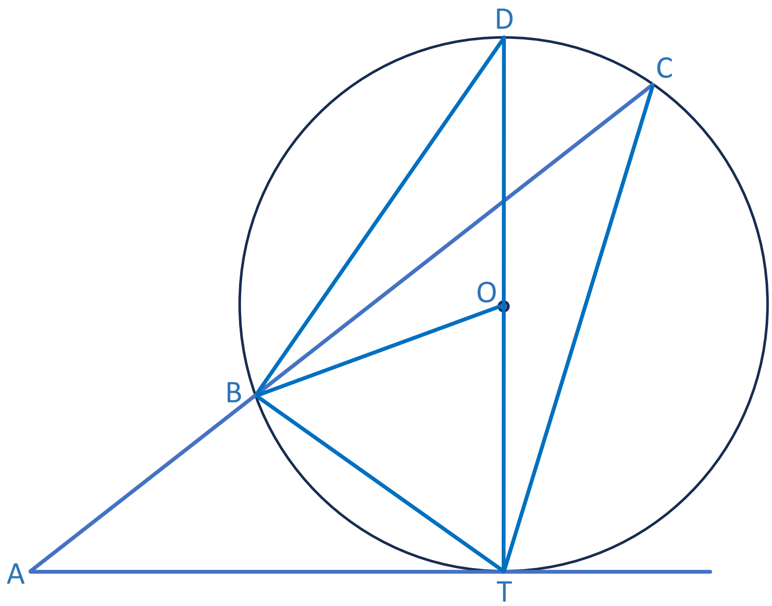
so these are trivial extensions of the case we have proved already.

Harder

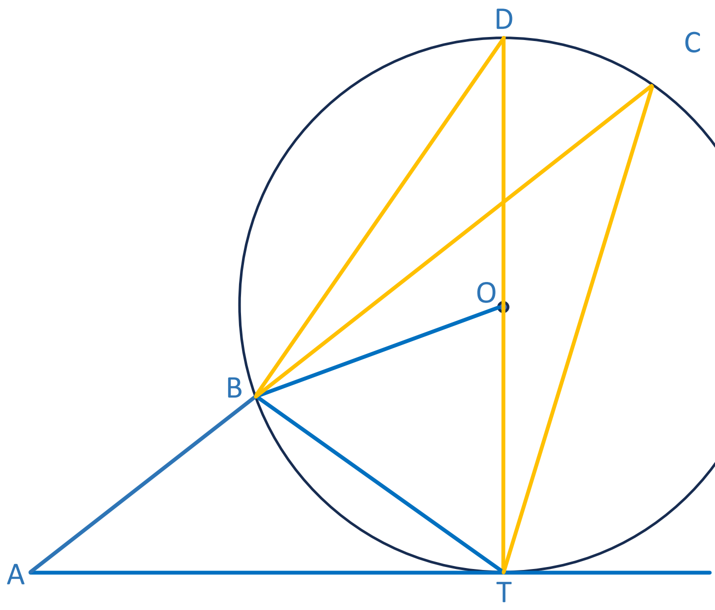
Prove the tangent-secant theorem, i.e. that the lengths AT, AB and AC satisfy $AT^2 = AB.AC$



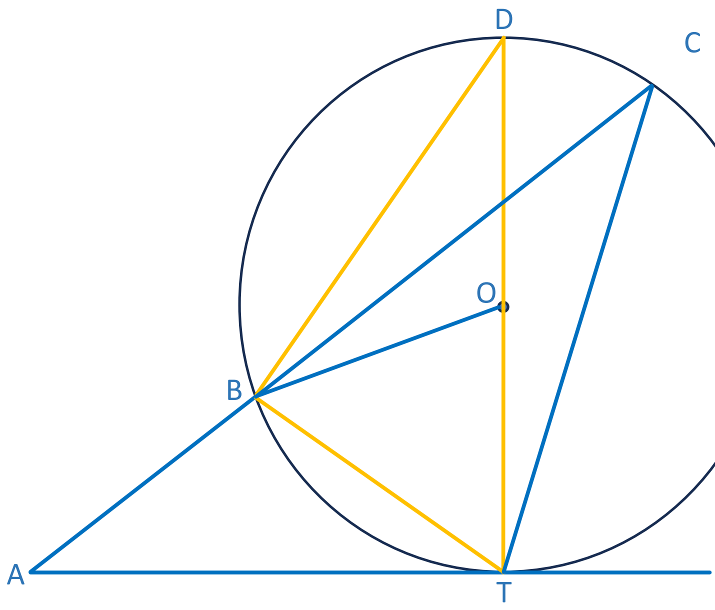
The way to solve this is to construct the perpendicular from T through the origin of the circle to the other side to D, and then complete the triangles as in the diagram.



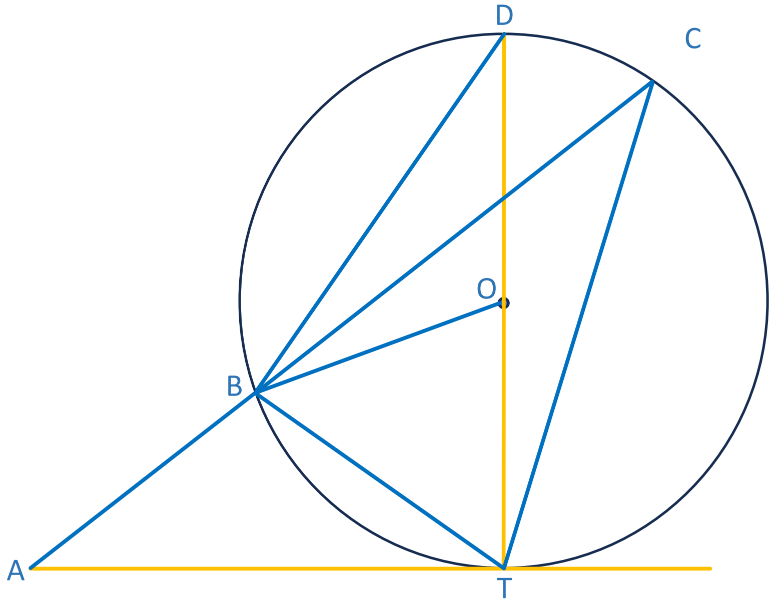
The first stage is then to notice that ∠TCB = ∠TDB because both of these are half ∠TOB, using the theorem in the easier question.



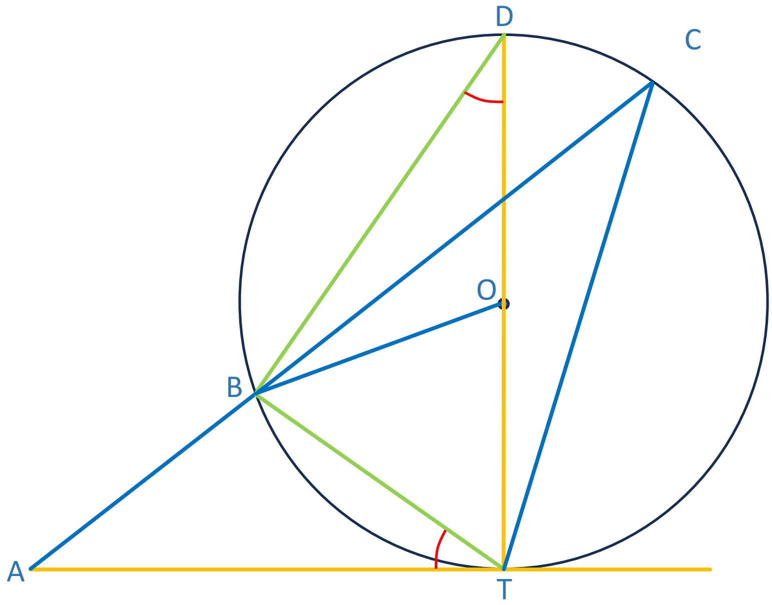
The second stage is to note that the triangle BDT is a right angle triangle with the right angle at B. This is because ∠TBD is half of ∠TOD which, in turn (because TOD is a straight line) is 180°.



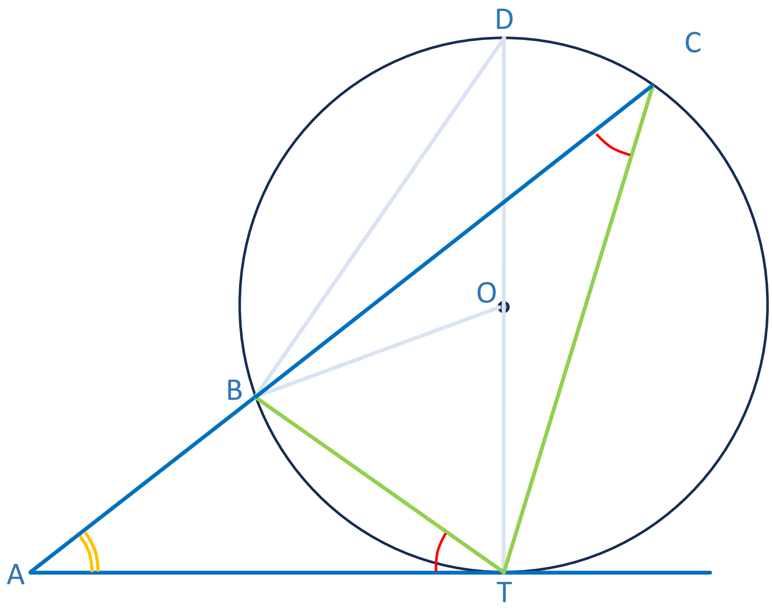
Third we note that ∠ATD is a right angle because the line TD is perpendicular to AT.



Fourth we can now calculate ∠ATB: ∠ATB + ∠BTD is 90°; and ∠BTD + ∠TDB is 90°, so ∠ATB = ∠TDB, which from the first stage we know equals ∠TCB. Hence ∠ATB = ∠TCB.



Consider now the triangles ABT and ATC. They share the angle at A and ∠ATB = ∠ACT; so these triangles are similar.



This means that AT/AB = AC/AT; hence AB.AC = AT².