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Easier

In a game of chance where you are quoted odds of evens*, e.g. placing your chips on either Red or Black in roulette, the amount you win is the amount you bet - you also get your stake back. (Note that the chance of winning is *not* the same as the odds: the odds define what the casino or bookmaker will pay if you win.)

Show that, with deep enough pockets, you can always win by playing 'double or quit', and also that this doesn't actually depend on the chance of winning (provided there *is* a chance of winning).

How much do you win each time you quit?

* Try not to think about that too hard!

Assume we start with a bet of £1:

If we win we get £2 back (including the stake). If we lose then we double our stake and bet £2.

If we then win we get £4 back, but the total we have bet is £1 (lost) + £2 (win) i.e. £3. If we lose we double our stake to £8…

Putting this in a table like the one below, we progress down the table until we win. If we want we can then start over again with a stake of £1.

Stake	Amount we get if we win (including stake)	Total amount we have bet so far	Profit if we win
£1	£2	£1	£2-£1=£1
£2	£4	£1+£2=£3	£4-£3=£1
£4	£8	£1+£2+£4=£7	£8-£7=£1
£8	£16	£1+£2+£4+£8=£15	£16-£15=£1
⋮	⋮	⋮	⋮
£2^N	£2^{N+1}	£2^{N+1}-£1	£1

Looking at the table we can see that we always make a profit of £1 when we quit.

Notice that this doesn't depend on the chance of winning, which could be very low, it only depends on the odds and playing double or quits.

Note that this strategy will fail in a normal casino as there will be a limit on the size of bet that can be placed.

Harder

You change your strategy so that instead of doubling your stake increases linearly, i.e. the stake is £1, £2, £3 £4, etc.

Assuming the chance of winning is p and the odds are $(w+1):1$, will this strategy ever make you money?

The table that was produced for the easier problem can be modified slightly to produce the numbers we want. We only have to include the probability that you stop after a given number of games:

Stake	Amount we get if we win (including stake)	Total amount we have bet so far	Profit if we win	Chance we quit at this point
£1	£(w+1)	£1	£W	p
£2	£2(w+1)	£1+£2=£3	£(2W-1)	$p(1-p)$
£3	£3(w+1)	£1+£2+£3=£6	£(3W-3)	$p(1-p)^2$
£4	£4(w+1)	£1+£2+£3+£4=£10	£(4W-6)	$p(1-p)^3$
⋮	⋮	⋮	⋮	
£N	£N(w+1)	£N(N+1)/2	£(NW-N(N-1)/2)	$p(1-p)^{N-1}$

From this we can see that our expected profit is

$$T = \sum_{N=1} \left(wN - \frac{N(N-1)}{2} \right) p(1-p)^{N-1}$$

There are two complicated sums to be done, but they look more familiar if we take out some constant factors:

$$T = wp\sum_{N=1} N(1-p)^{N-1} - \frac{p(1-p)}{2}\sum_{N=1} N(N-1)(1-p)^{N-2}$$

We now see that the factors we need are

$$\sum_{N=1} N(1-p)^{N-1}$$

and

$$\sum_{N=1} N(N-1)(1-p)^{N-2}$$

If we temporarily write $q = (1-p)$ we have [pay attention to the limits]

$$\sum_{N=1} N(1-p)^{N-1} = \sum_{N=1} Nq^{N-1} = \frac{d}{dq}\sum_{N=0} q^N = \frac{d}{dq}\left(\frac{1}{1-q}\right) = \frac{1}{(1-q)^2} = \frac{1}{p^2}$$

And using the same observation:

$$\sum_{N=1} N(N-1)(1-p)^{N-2} = \frac{d^2}{dq^2}\left(\frac{1}{1-q}\right) = \frac{2}{p^3}$$

This means our expected winnings are:

$$\begin{aligned} T &= wp\frac{1}{p^2} - \frac{p(1-p)}{2}\frac{2}{p^3} \\ &= \frac{(w+1)p-1}{p^2} \end{aligned}$$

So we would make money if the odds are such that $(w+1)p > 1$. This is, however, a highly unlikely event as the casino or bookmaker will adjust the odds so that this doesn't happen: it would imply that they expect to make a loss on every bet.